

# Entanglement cost of two-qubit orthogonal measurements

Somshubhro Bandyopadhyay

*Department of Physics and Center for Astroparticle Physics and Space Science,  
Bose Institute, Block EN, Sector V, Kolkata 700091, India\**

Ramij Rahaman<sup>†</sup>

*Department of Informatics, University of Bergen, PB-7803, Bergen-5020, Norway*

William K. Wootters

*Department of Physics, Williams College, Williamstown, MA 01267, USA  
Department of Applied Physics, Kigali Institute of Science and Technology, B.P. 3900, Kigali, Rwanda<sup>‡</sup>*

**Abstract.** The “entanglement cost” of a bipartite measurement is the amount of shared entanglement two participants need to use up in order to carry out the given measurement by means of local operations and classical communication. Here we numerically investigate the entanglement cost of generic orthogonal measurements on two qubits. Our results strongly suggest that for almost all measurements of this kind, the entanglement cost is strictly greater than the average entanglement of the eigenstates associated with the measurements, implying that the nonseparability of a two-qubit orthogonal measurement is generically distinct from the nonseparability of its eigenstates.

Certain measurements on composite systems, whose parts are spatially separated, cannot be implemented by local operations and classical communication (LOCC). Entangling measurements certainly belong to this category, and the so-called “non-locality without entanglement” shows that even some unentangled measurements cannot be performed by LOCC alone [1]. The latter example clearly exhibits a kind of nonseparability associated with joint quantum measurements that is not entirely captured by the entanglement of their eigenstates.

It is, however, possible to quantify this nonseparability by computing the amount of entanglement that is required to perform a measurement on spatially separated subsystems by LOCC. This amount is referred to as the “entanglement cost” of the given measurement, the precise definition of which is given below.

We imagine two parties, Alice and Bob, each holding one of the two particles to be measured. They are allowed to do any sequence of LOCC but are not allowed to transmit quantum information. Rather we give them, as a resource, shared entangled pure states (whose form Alice and Bob are allowed to choose), and we keep track of the amount of entanglement they spend in performing the measurement. For a pure state  $|\psi\rangle_{AB}$  of a bipartite

system  $AB$ , the entanglement is

$$\mathcal{E}(|\psi\rangle_{AB}) = -\text{Tr}(\rho_A \log \rho_A), \quad (1)$$

where  $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$  is the reduced density matrix of the subsystem  $A$ . The logarithm is always taken with respect to base two, so the entanglement is measured in ebits. If Alice and Bob completely use up a copy of a state  $|\eta\rangle$ , then the entanglement cost of that operation is simply  $\mathcal{E}(|\eta\rangle)$ . On the other hand, if they convert an entangled state into a less entangled state, then the cost is the difference, that is, the amount of entanglement lost. It should be noted that a different notion of the entanglement cost is considered in Ref. [2], namely the amount of entanglement needed to effect a Naimark extension of a given POVM.

Computing the exact entanglement cost of a generic bipartite measurement seems to be a hard problem except in special cases. These cases include any maximally entangled measurement in  $d \otimes d$ , which costs exactly  $\log_2 d$  ebits (a particular implementation achieving this cost uses a maximally entangled state to teleport the information in one of the two parts to the location of the other), and any complete measurement in  $d \otimes d$  which is invariant under all local Pauli operations, for which the cost is equal to the average entanglement of the states associated with the outcomes.

Notwithstanding the difficulty of computing the exact entanglement cost of measurements, some progress has been made in obtaining lower and upper bounds [3–9]. A general lower bound can be computed by considering the entanglement production capacity of measurements [10–12]. In this way one can show that for any complete measurement on a bipartite system the entanglement cost is at least as great as the average entanglement of the pure states associated with the outcomes. We call this lower bound the “entropy bound,” since the entanglement of a pure bipartite state is quantified by the entropy of either of the two parts. This bound is achieved for the special classes of measurements mentioned in the preceding paragraph.

Interestingly, for a certain class of orthogonal partially entangling measurements on two qubits, the entanglement cost has been shown to be strictly greater than the average entanglement of the eigenstates [3], suggesting

\*Electronic address: som@bosemain.boseinst.ac.in

<sup>†</sup>Electronic address: Ramij.Rahaman@ii.uib.no

<sup>‡</sup>Electronic address: William.K.Wootters@williams.edu

that this feature is perhaps a generic property of measurements. If it is, then one can say that the nonseparability of a measurement is generically a distinct property from the nonseparability of its eigenstates. In this paper we investigate, by means of numerical calculations, whether this is the case for complete orthogonal measurements on two qubits.

Our numerical results strongly indicate that for generic orthogonal two-qubit measurements, entanglement cost is indeed strictly greater than the average entanglement of the measurement eigenstates; that is, it is strictly greater than the entropy bound. We reach this conclusion by computing another lower bound, again based on entanglement production but with a more refined analysis than the one leading to the entropy bound. We evaluate this bound for a broad sample of orthogonal measurements, covering fairly densely the range of possible values of the average entanglement of the eigenstates. Though there are exceptional cases for which the more refined bound is equal to the entropy bound, it appears that the set of such cases is very small, probably constituting a manifold of lower dimension than the manifold of all orthogonal measurements. The exceptional cases that we can explicitly identify are the following: (i) any measurement in a product basis (for which the entanglement cost is zero), (ii) any measurement whose eigenstates are all maximally entangled (for which the entanglement cost is 1 ebit), and (iii) any measurement that is equivalent, under local unitaries, to the measurement with eigenstates  $\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), |01\rangle, |10\rangle \right\}$ .

A complete orthogonal projective measurement  $M$  is specified by a collection of rank-one projection operators  $\Pi_i = |\psi_i\rangle\langle\psi_i|$  that sum to the identity. Such operators are necessarily orthogonal; that is,  $\text{Tr}(\Pi_i\Pi_j) = \delta_{ij}$ . In this paper we understand  $M$  to provide only a rule for computing probabilities, not for computing the final state of the system after measurement. So a realization of  $M$  can be any procedure that yields the correct probabilities, even if, for example, it destroys the measured system. We allow the possibility of probabilistic measurement procedures, in which the probabilities might depend on the initial state of the system being measured. As we will be computing an average cost, we need to specify an initial state in order for the average to be well defined. We assume that this initial state is the completely mixed state, which we regard as the least biased choice. We now give the formal definition of entanglement cost associated with a quantum measurement.

Given a measurement  $M$ , let  $\mathcal{P}(M)$  be the set of all LOCC procedures  $P$  such that (a)  $P$  uses pure entangled states and LOCC and (b)  $P$  realizes  $M$  exactly. Then  $C(M)$ , the entanglement cost of measurement  $M$ , is defined to be

$$C(M) = \inf_{P \in \mathcal{P}(M)} \langle \mathcal{E}_{\text{initial}} - \mathcal{E}_{\text{final}} \rangle \quad (2)$$

where  $\mathcal{E}_{\text{initial}}$  is the total entanglement of all the resource states used in the procedure,  $\mathcal{E}_{\text{final}}$  is the distillable en-

tanglement of the state remaining at the end of the procedure, and  $\langle \dots \rangle$  indicates an average over all possible results of  $P$ , when the system on which the measurement is being performed is initially in the completely mixed state.

As was mentioned above, our lower bound on the entanglement cost is obtained by considering the entanglement production capacity of  $M$ . That this capacity is a lower bound on the entanglement cost of  $M$  follows from the fact that in performing the measurement, the participants must consume at least as much entanglement as the measurement can produce, since entanglement cannot increase, on an average, by LOCC. Specifically we imagine that in addition to qubits  $A$  and  $B$  on which Alice and Bob want to perform their measurement, they also have in their possession auxiliary qubits  $C$  and  $D$ . (Here  $A$  and  $C$  are held by Alice, and  $B$  and  $D$  are held by Bob.) Now consider an initial state of four qubits such that the measurement  $M$  on qubits  $A$  and  $B$  collapses qubits  $C$  and  $D$  into a possibly entangled state. Then, the average amount by which the measurement increases the entanglement between Alice and Bob is a lower bound on  $C(M)$ . That is,

$$C(M) \geq C_L(M) = \overline{\mathcal{E}}_{CD} - \mathcal{E}_{AC:BD}, \quad (3)$$

where,  $\overline{\mathcal{E}}_{CD}$  is the average final entanglement between the qubits  $C$  and  $D$  and  $\mathcal{E}_{AC:BD}$  is the initial entanglement across the bipartition  $AC : BD$ . We do not consider any final entanglement between  $A$  and  $B$ , because we want our lower bound to apply to any procedure that implements  $M$ , even if it destroys  $A$  and  $B$ .

The initial state is chosen to be

$$|\chi\rangle_{ABCD} = \frac{1}{2} \sum_{i=1}^4 |\psi_i\rangle_{AB} \otimes |\phi_i\rangle_{CD} \quad (4)$$

where the states  $\{|\psi_i\rangle, i = 1, \dots, 4\}$  are the orthogonal eigenstates of the measurement  $M$  that Alice and Bob want to perform. The states  $\{|\phi_i\rangle, i = 1, \dots, 4\}$  are what we will call the “detector” states, so named because they correspond to the measurement outcomes on the system  $AB$ . While, for a given measurement  $M$ , its eigenstates are fixed, the detector states can be completely arbitrary, except, we insist that they be mutually orthogonal. This restriction guarantees that the initial state of  $AB$  is the completely mixed state, in accordance with our definition of entanglement cost. For any particular choice of the detector states, a lower bound on  $C(M)$  is given by the formula

$$C(M) \geq C_L(M) = \frac{1}{4} \sum_{i=1}^4 E(|\phi_i\rangle) - \mathcal{E}_{AC:BD}. \quad (5)$$

The above quantity can be maximized numerically over the detector states to obtain the best absolute lower bound as has been done in Ref. [3] for a class of two-qubit orthogonal measurements. However, here our objective is not to get the best possible lower bound for

$M$ . We only want to find out whether the lower bound is typically greater than the entropy bound. We therefore consider the quantity

$$\delta = C_L(M) - \frac{1}{4} \sum_{i=1}^4 E(|\psi_i\rangle) \quad (6)$$

and maximize it only over a discrete grid of detector states. If the resulting maximum is unambiguously positive, that is, greater than the numerical error, we can conclude that for the given measurement  $M$ , the entanglement cost is strictly greater than the entropy bound.

For a generic orthogonal two-qubit measurement  $M$  its eigenstates are simply a set of orthogonal vectors  $\{|\psi_i\rangle, i = 1, 2, 3, 4\}$  in  $2 \otimes 2$ . The most general canonical form of four orthogonal states in  $2 \otimes 2$ , up to local unitaries, can be obtained in the following fashion. Walgate *et al.* [13] have shown that any pair of orthogonal states of two qubits can, by local rotations, be brought to the form  $\{\alpha|00\rangle + \beta|11'\rangle, \gamma|01\rangle + \delta|10'\rangle\}$ , where  $|0'\rangle$  and  $|1'\rangle$  constitute an orthogonal basis for the second qubit. Thus we can write the first two eigenstates of our measurement as

$$|\psi_1\rangle = \cos(a)|00\rangle + e^{ib}\sin(a)|1\rangle(\cos(u)|0\rangle + e^{iv}\sin(u)|1\rangle); \quad (7)$$

$$|\psi_2\rangle = \cos(c)|01\rangle + e^{id}\sin(c)|1\rangle(e^{-iv}\sin(u)|0\rangle - \cos(u)|1\rangle). \quad (8)$$

(Since the quantities of interest are independent of the overall phases of the state vectors, we are free to take the coefficient of the first term in each of these expressions to be real and non-negative.) We now define two states that are orthogonal to both  $|\psi_1\rangle$  and  $|\psi_2\rangle$ :

$$|\psi_1^\perp\rangle = e^{-ib}\sin(a)|00\rangle - \cos(a)|1\rangle(\cos(u)|0\rangle + e^{iv}\sin(u)|1\rangle); \quad (9)$$

$$|\psi_2^\perp\rangle = e^{-id}\sin(c)|01\rangle - \cos(c)|1\rangle(e^{-iv}\sin(u)|0\rangle - \cos(u)|1\rangle). \quad (10)$$

The remaining two eigenstates of the measurement will be linear combinations of  $|\psi_1^\perp\rangle$  and  $|\psi_2^\perp\rangle$ :

$$|\psi_3\rangle = \cos(x)|\psi_1^\perp\rangle + e^{iy}\sin(x)|\psi_2^\perp\rangle; \quad (11)$$

$$|\psi_4\rangle = e^{-iy}\sin(x)|\psi_1^\perp\rangle - \cos(x)|\psi_2^\perp\rangle. \quad (12)$$

We cover the full set of orthogonal measurements by allowing the following ranges for the parameters:  $0 \leq a, c, u, x \leq \pi/2$ ,  $0 \leq b, d, v, y \leq 2\pi$ .

To generate the specific subset of orthogonal measurements to be considered in our numerical calculation, we step through the ranges of the parameters, using the following step sizes: for  $a, c$ , and  $u$ , step size  $\pi/24$ ; for  $b, d, v$ , and  $y$ , step size  $\pi/12$ ; for  $x$ , step size  $\pi/16$ . For any given measurement  $M$ , we parameterize the detector states  $\{|\phi_i\rangle, i = 1, \dots, 4\}$ , in the same way using a different set of parameters, and use step sizes half as large as

those we use for the measurement states. In each case, that is, for each measurement, we record the largest value of  $\delta$  obtained by stepping through the discrete grid of detector states.

Our results are plotted in Fig. 1. (The plot includes some additional points not covered by the grid described above. These points were chosen more or less arbitrarily.) For almost every measurement we considered, the refined lower bound is strictly larger than the entropy bound. The only exceptions we have found are those mentioned earlier: the cases of product-state or maximally entangled measurements, and the one intermediate case, with eigenstates  $\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)|01\rangle, |10\rangle \right\}$ .

Our results strongly suggest that for almost all orthogonal two-qubit measurements, the lower bound on the entanglement cost is strictly greater than the average entanglement of the states themselves. Thus the nonseparability associated with a measurement appears to generically exceed the nonseparability of its eigenstates. We note that many of the points plotted in the lower half of Fig. 1 lie very close to the entropy bound. This fact suggests that there may be a class of special measurements, which we have not yet identified explicitly, for which our lower bound, even when maximized over all detector states, is exactly equal to the entropy bound. It would be interesting to identify such measurements if they do indeed exist.

Note that the lower bounds are also valid asymptotically. Suppose  $N$  pairs of qubits are given to Alice and Bob and they are to perform the same measurement on each pair. It is conceivable that by performing a measurement involving all  $N$  pairs they can achieve better efficiency. Even in this setting the lower bounds obtained are still applicable. To see this, imagine that each pair is initially entangled with a pair of auxiliary qubits. Since both the initial and final entanglements of the whole system across the bipartition  $AC : BD$  are proportional to  $N$ , the original argument still holds.

Several open questions remain. The method presented here and in Ref.[3] works reasonably well for obtaining lower bounds, although in higher dimensions numerical calculations could become much more complex. Obtaining *upper bounds*, even for two-qubit measurements, is still very much an open question. Only for special classes of orthogonal measurements have non-trivial upper bounds been obtained, and even then only for a particular range of entanglement of the eigenstates. Pursuing these questions should shed more light on the nature of nonseparability in quantum measurements.

### Acknowledgments

R.R. would like to acknowledge thankfully the financial support of the Norwegian Research Council.

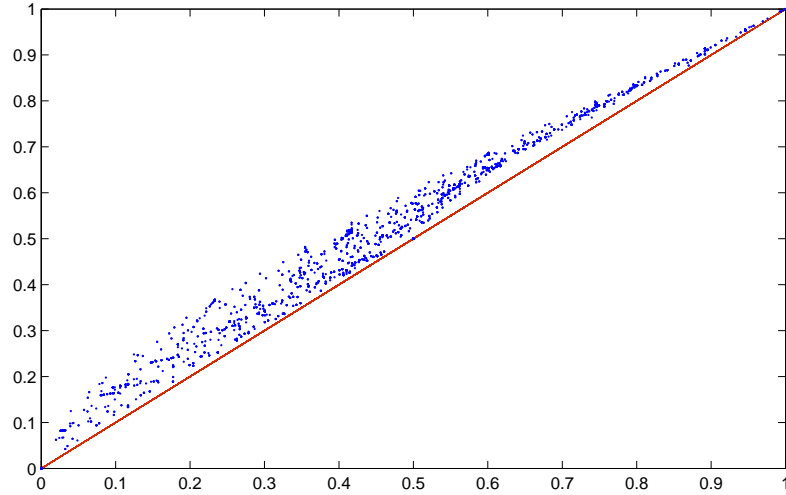


Figure 1: Blue dots: Plot of the lower bound of the entanglement cost  $C_L(M)$  with respect to the entropy bound for general two-qubit orthogonal measurements. Red line: Plot of the entropy bound.

- 
- [1] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **59**, 1070 (1999).
  - [2] R. Jozsa, M. Koashi, N. Linden, S. Popescu, S. Presnell, D. Shepherd, and A. Winter, *Quantum Inf. Comput.* **3**, 405 (2003).
  - [3] S. Bandyopadhyay, G. Brassard, S. Kimmel, and W. K. Wootters, *Phys. Rev. A*, **80**, 012313 (2009).
  - [4] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, *IEEE Trans. Inf. Theory* **49**, 1895 (2003).
  - [5] A. Cheffles, C. R. Gilson, and S. M. Barnett, *Phys. Rev. A*, **63**, 032314 (2001).
  - [6] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, *Phys. Rev. Lett.*, **86**, 544 (2001).
  - [7] D. W. Berry, *Phys. Rev. A*, **75**, 032349 (2007).
  - [8] D. Jonathan and M. B. Plenio, *Phys. Rev. Lett.*, **83**, 1455 (1999).
  - [9] G. Vidal, *Phys. Rev. Lett.*, **83**, 1046 (1999).
  - [10] J. A. Smolin, *Phys. Rev. A*, **63**, 032306 (2001).
  - [11] S. Ghosh, G. Kar, A. Roy, A. Sen(De) and U. Sen, *Phys. Rev. Lett.* **87**, 277902 (2001).
  - [12] M. Horodecki, A. Sen(De), U. Sen, and K. Horodecki, *Phys. Rev. Lett.*, **90**, 047902 (2003).
  - [13] J. Walgate, A. J. Short, L. Hardy and V. Vedral *Phys. Rev. Lett.*, **85**, 4972 (2000).